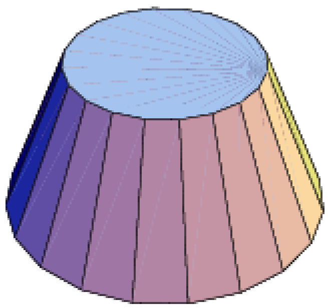
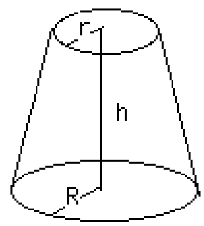
Conical Frustum

A Conical Frustum is a Frustum created by slicing the top off a cone (with the cut made parallel to the base), forming a lower base and an upper base that are circular and parallel. Let h be the height, R the radius of the lower base, and r the radius of the upper base as pictured below:



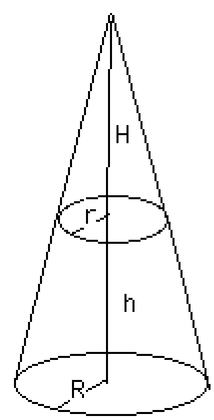


The Volume of Frustum:

The Volume of the Frustum could be found using the formula: $V = \frac{\pi h}{3}(R^2 + Rr + r^2)$

Now, let's derive the formula without using calculus.

Consider the cone before it was cut. Let the height of the cut be H.



The volume of the cone, $V = \frac{1}{3}\pi r^2 h$

The volume of the original pre-cut cone , $V_o = \frac{1}{3}\pi R^2(H + h)$

The volume of the cut part, $V_c = \frac{1}{3}\pi r^2 H$

Now, to get the volume of the frustum (V_F), we have to subtract the volume of the cut part from the volume of the original cone. So,

$$V_F = V_o - V_c$$

$$= \frac{1}{3}\pi R^2(H + h) - \frac{1}{3}\pi r^2 H$$

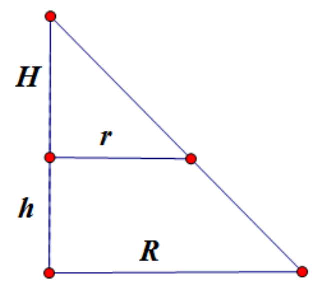
$$= \frac{1}{3}\pi R^2 H - \frac{1}{3}\pi R^2 h - \frac{1}{3}\pi r^2 H$$

$$= \frac{1}{3}\pi(R^2H + R^2h - r^2H)$$

$$= \frac{1}{3}\pi\left(R^2h + H(R^2 - r^2)\right)$$

$$= \frac{1}{3}\pi\left(R^2h + H(R - r)(R + r)\right) \ldots\ldots\ldots (i)$$

Now, consider the original pre-cut cone. The triangles formed by the height and the bases are similar by AA similarity and the sides are proportional.



Hence, $\frac{H}{r} = \frac{H+h}{R} \ldots\ldots\ldots (ii)$

So we have,

$HR = r(H + h)$

$\Rightarrow HR = Hr + hr$

$\Rightarrow HR - Hr = hr$

$\Rightarrow H(R - r) = hr$

Substitute $H(R - r)$ in equation (i), we have

$$V_F = \frac{1}{3}\pi[R^2h + hr(R + r)]$$

$$= \frac{1}{3}\pi h[R^2 + r(R + r)]$$

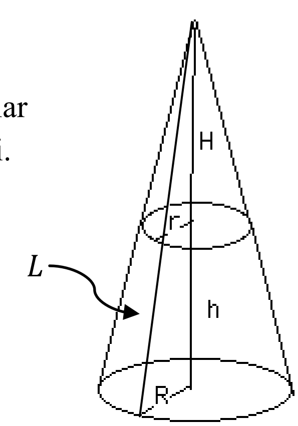
$$= \frac{1}{3}\pi h[R^2 + Rr + r^2]$$

Hence the formula of volume of the Frustum.

Total Surface Area of a Frustum:

Now, let's examine how to find the Surface Area of the frustum.

We know the lateral area of a right circular cone is πrs. For the right circular cone here, let L be the *slant height* and r *and* R be the top and bottom radii. Then,



$$L = \sqrt{(H + h)^2 + R^2}$$

So, the lateral area of the pre-cut cone, $LA_o = \pi r\sqrt{(H+h)^2 + R^2}$

The lateral area of the cut part, $LA_c = \pi r\sqrt{H^2 + r^2}$

Hence, area of lateral fractum, $LA_F = LA_o - LA_c = \pi r\sqrt{(H+h)^2 + R^2} - \pi r\sqrt{H^2 + r^2}$

To find the toatal surface area of the frusum, we need to add the area o the base. Additionally, ther area of the base is the area of circle with radius r.

Hence, the total surface area of a frustum, $SA_F = \pi r\sqrt{(H+h)^2 + R^2} - \pi r\sqrt{H^2 + r^2} + \pi r^2$.

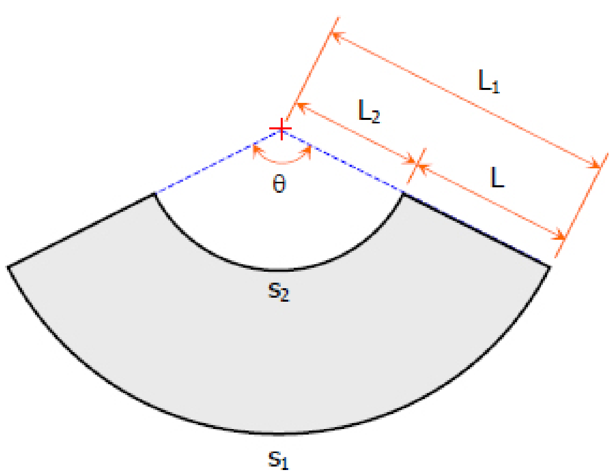
Lateral Area of a Frustum of a Right Circular Cone:

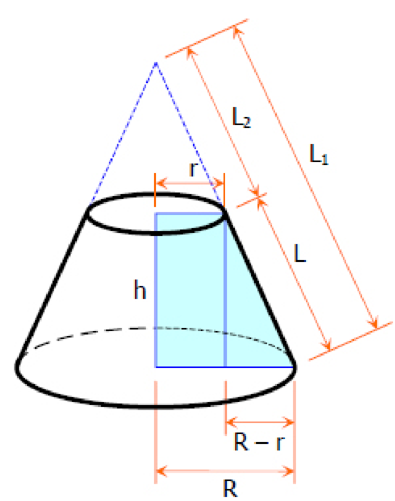
The lateral area of a frustum of a right circular cone is given by

$$A = \pi(R + r)L$$

Where, R = radius of the lower base, r = radius of the upper base, and L = the length of the lateral side.

Now, the lateral area of right circular cone is the difference of the areas of sections of a circle of radii L_1 and L_2, and common central angle θ (as pictured below):



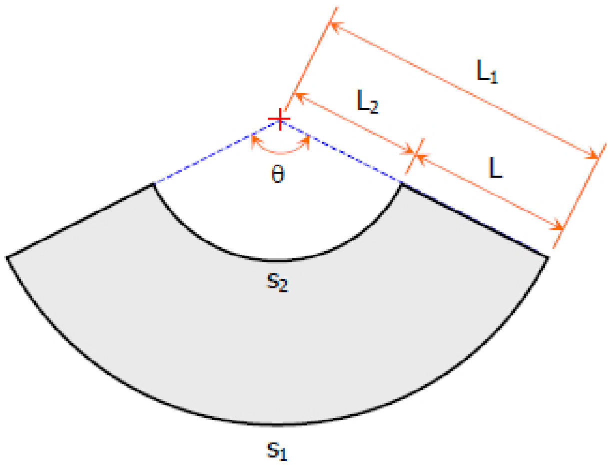


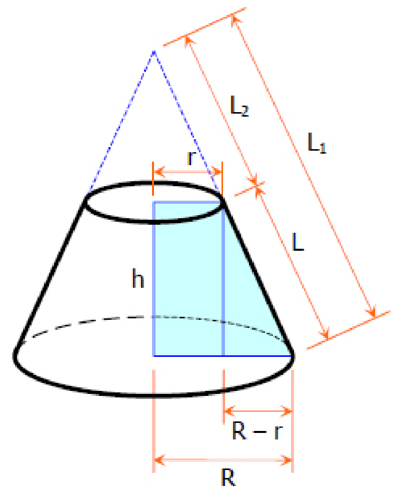
By ratio and proportion:

$$\frac{L_1}{R} = \frac{L}{R - r}$$

$$L_1 = \frac{RL}{R - r}$$

And, from the figures:





$$L_2 = L_1 - L$$

$$= \frac{RL}{R-r} - L$$

$$= \frac{RL - (R-r)L}{R-r}$$

$$= \frac{rL}{R-r}$$

The length of arc is the circumfence of the base:

$$s_1 = 2\pi R$$

$$s_2 = 2\pi r$$

Again, from the figure:

$$A = \frac{1}{2}s_1 L_1 - \frac{1}{2}s_2 L_2$$

$$= \frac{1}{2}(2\pi R)\left(\frac{RL}{R-r}\right) - \frac{1}{2}(2\pi r)\left(\frac{rL}{R-r}\right)$$

$$= \frac{\pi R^2 L}{R-r} - \frac{\pi r^2 L}{R-r}$$

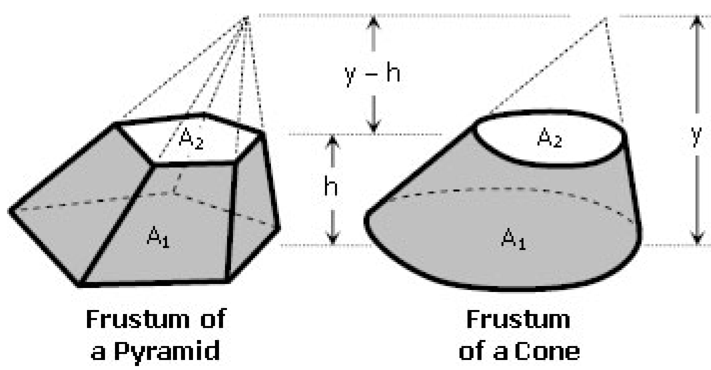
$$= \frac{\pi R^2 L - \pi r^2 L}{R - r}$$

$$= \frac{\pi (R^2 - r^2) L}{R - r}$$

$$= \frac{\pi (R - r)(R + r) L}{R - r}$$

Hence, $A = \pi(R + r)L$

Volume of a truncated Pyramid with a square base:



Let, A_1 = the area of the lower base

A_2 = the area of the upper base

h = perpendicular distance between A_1 and A_2 (also known as the atitude of frustum)

Note here that A_1 and A_2 are parallel to each other.

Now, Volume of frustum, $V_1 = \frac{1}{3} A_1 y$

And top of cone, $V_2 = \frac{1}{3} A_2 (y - h)$

So, $V = V_1 - V_2$

$$= \frac{1}{3} A_1 y - \frac{1}{3} A_2 (y - h)$$

$$= \frac{1}{3}A_1 y - \frac{1}{3}A_2 y + \frac{1}{3}A_1 h$$

$$= \frac{1}{3}[(A_1 - A_2)y + A_2 h] \dots \dots \dots (i)$$

By similar solids rule,

$$\frac{A_2}{A_1} = \left(\frac{y-h}{y}\right)^2$$

$$\sqrt{\frac{A_2}{A_1}} = 1 - \frac{h}{y}$$

$$\frac{h}{y} = 1 - \sqrt{\frac{A_2}{A_1}}$$

$$= \frac{\sqrt{A_1} - \sqrt{A_2}}{\sqrt{A_1}}$$

So,

$$\frac{y}{h} = \frac{\sqrt{A_1}}{\sqrt{A_1} - \sqrt{A_2}}$$

$$y = h\frac{\sqrt{A_1}}{\sqrt{A_1} - \sqrt{A_2}}$$

$$= h\frac{\sqrt{A_1}}{\sqrt{A_1} - \sqrt{A_2}} \cdot \left(\frac{\sqrt{A_1} + \sqrt{A_2}}{\sqrt{A_1} + \sqrt{A_2}}\right)$$

$$y = \frac{A_1 + \sqrt{A_1 A_2}}{A_1 - A_2}h$$

Now, substituting y in equation (i),

$$V = \frac{1}{3}\left[(A_1 - A_2)\left(\frac{A_1 + \sqrt{A_1 A_2}}{A_1 - A_2}h\right) + A_2 h\right]$$

$$V = \frac{1}{3}\left[\left(A_1 + \sqrt{A_1 A_2}\right)h + A_2 h\right]$$

$$V = \frac{h}{3}\left(A_1 + A_2 + \sqrt{A_1 A_2}\right)$$